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Explicit formulae for the powers of a Schrödinger-like ordinary differential operator

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Abstract. We call $L = \frac{\mathrm{d}^r}{\mathrm{d}x^r} + u$ a Schrödinger-like differential operator and derive some explicit formulae for the powers or iterations L^n $(n = 1, 2, \ldots)$ of it.

1. Introduction

It is desirable for several purposes to have an explicit formula for the powers or iterations $L^n = LL \ldots L$ (n times) of an ordinary linear differential operator of the simple form

$$L = D^r + u \tag{1}$$

where $D = \frac{\mathrm{d}}{\mathrm{d}x}$ and $u = u(x)$ is a function. There is one independent variable x, while the number N of dependent variables is arbitrary. Thus, we assume $u = u(x)$ to be $N \times N$ matrix valued. The scalar case is included with $N = 1$. For some low values of r such operators have many mathematical and physical applications. The case $r = 1$ is important since every linear (autonomous) evolution equation can be brought into the form

$$(D + u)y = f. \tag{2}$$

The case $r = 2$ concerns the time-independent one-dimensional Schrödinger operator

$$L = D^2 + u. \tag{3}$$

There is an abundance of applications and ramifications of the latter. In particular, let us mention the relation, via the inverse scattering method, to the Korteweg–de Vries (KdV) equation

$$u_t = 6uu_x + u_{xxx} \tag{4}$$

or, more generally, to the KdV hierarchy of solitonic partial differential equations

$$u_t = \frac{\partial}{\partial x} G_n[u]. \tag{5}$$

The latter admits the Lax representation [1, 2]

$$L_t \equiv [A_n, L] \tag{6}$$

where L is the one-dimensional Schrödinger operator (3) and A_n is a linear differential operator of the order $2n + 1$, the coefficients of which are differential polynomials in u. In particular, we have for $n = 1$

$$A_1 = 4D^3 + 3uD + 3Du \tag{7}$$

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and (6) becomes the KdV equation (4).

Since equation (1) generalizes the one-dimensional Schrödinger operator (3), we propose to call it a *Schrödinger-like differential operator* and $u = u(x)$ the *potential*. For $r = 4$ there are also physical applications, for example, in the theory of elasticity. It is a challenge, from a mathematical point of view, to construct an explicit formula for L^n for general r, that is an expression for the coefficients of the linear differential operator L^n. This is what we shall do in this paper. Our investigation is partly inspired by Torriani's paper [3].

2. An explicit formula for L^n

Let us start with some general formulae for the powers of the sum of two linear operators.

Proposition 1. Let $A : V \to V, B : V \to V$ be two linear operators acting on a vector space V. There holds

$$(A+B)^n = \sum_{p=0}^{n} \sum_{n_0,n_1,\ldots,n_p} A^{n_0} B A^{n_1} B A^{n_2} \ldots B A^{n_p} \tag{8}$$

where the inner sum runs through all integers $n_0 \geqslant 0, n_1 \geqslant 0, \ldots, n_p \geqslant 0$ such that $n_0 + n_1 + \cdots + n_p = n - p$. For $p = 0$ the inner sum equals, by definition, the single term A^n.

Proof. There is a combinatorial argument behind the above, but we prefer to use a more transparent proof by means of mathematical induction. Let us set $L = A + B$. A simple induction shows

$$L^n = A^n + \sum_{m=1}^{n} L^{m-1} B A^{n-m}.$$

Initially we take this formula and then insert the following into it

$$L^{m-1} = A^{m-1} + \sum_{m_1=1}^{m-1} L^{m_1-1} B A^{m-1-m_1}.$$

Then into the latter formula we insert

$$L^{m_1-1} = A^{m_1-1} + \sum_{m_2=1}^{m_1-1} L^{m_2-1} B A^{m_1-1-m_2}$$

and so on. In the pth step, we have to treat

$$\sum_{m_0=1}^{n} \sum_{m_1=1}^{m_0-1} \cdots \sum_{m_{p=1}}^{m_{p-1}-1} L^{m_p-1} B A^{m_{p-1}-1-m_p} B \ldots B A^{n-m_0}$$

in order to reduce the powers of L. Owing to the condition

$$1 \leqslant m_p < m_{p-1} < \cdots < m < m_0 \leqslant n$$

the procedure stops at $p = n - 1$ where

$$m_{n-1} = 1 \qquad m_{n-2} = 2, \ldots, m_0 = n$$

and we omit the powers of L. Finally, the index transformation

$$n_0 = m_{p-1} \qquad n_1 = m_{p-1} - 1 - m_p \qquad n_2 = m_{p-2} - 1 - m_{p-1}, \ldots, n_p = n - m_0$$

gives the result. □

Next we apply our formula to $L = A + B \equiv D^r + u$. Multiplication by a function u is notationally identified with u.

Proposition 2. The powers of a Schrödinger-like operator $L = D^r + u$ are given by

$$L^n y = y_{rn} + \sum_{p=1}^{n} \sum_{n_0,n_1,\ldots,n_p} D^{rn_0}(uD^{rn_1}(uD^{rn_2}\ldots(uy_{rn_p})) \tag{9}$$

where the inner sum runs through all integers $n_0 \geqslant 0, n_1 \geqslant 0, \ldots, n_p \geqslant 0$ such that $n_0 + n_1 + \cdots + n_p = n - p$.

Proof. Insert $A = D^r$, $B = u$ into (8). □

We now arrive at out main result. We abbreviate $u_k := D^k u$ for $k = 1, 2, \ldots, u_0 := u$.

Theorem 1. The nth power of an rth-order Schrödinger-like operator $L = D^r + u$ is given by

$$L^n y = y_{rn} + \sum_{p=1}^{n} \sum_{k_0,k_1,\ldots,k_p} C_{k_1 k_2 \ldots k_p} u_{k_1} u_{k_2} \ldots u_{k_p} y_{k_0} \tag{10}$$

where the inner sum runs through all integers $k_0, k_1, \ldots, k_p$ such that $k_0, k_1, \ldots, k_p \geqslant 0, k_0 + k_1 + \cdots + k_p = r(n - p)$, and where the coefficients are equal to

$$C_{k_1,k_2\ldots k_p} = \sum_{m_0,m_1,\ldots,m_p} \binom{rm_0}{k_1}\binom{rm_1 - k_1}{k_2}\binom{rm_2 - k_1 - k_2}{k_3} \ldots \binom{rm_p - k_1 - \cdots - k_{p-1}}{k_p}. \tag{11}$$

Here the sum runs through all integers $m_0, m_1, \ldots, m_p$ such that $0 \leqslant m_0 \leqslant m_1 \leqslant \cdots \leqslant m_p = n - p$.

Proof. We apply the iterated Leibniz rule to (9). Namely in the first step we insert

$$D^{rn_0}(uZ_1) = \sum_{k_1 \geqslant 0} \binom{rn_0}{k_1} u_{k_1} D^{rn_0 - k_1} Z_1$$

and in the second step we set

$$D^{rn_0 - k_1} Z_1 = D^{r(n_0+n_1)-k_1}(uZ_2) = \sum_{k_2 \geqslant 0} \binom{r(n_0 + n_1) - k_1}{k_2} D^{r(n_0+n_1)-k_1-k_2} Z_2$$

with suitable abbreviations Z_1, Z_2, and so on. Note that the sum over $k_1 \geqslant 0$ effectively ends at $k_1 = rn_0$ and the sum over $k_2 \geqslant 0$ ends at $k_2 = r(n_0+n_1)-k_1$, because the binomial coefficients vanish for higher values of k_1 or k_2. The final step of this mathematical induction reads

$$D^{r(n_0+n_1+\cdots+n_{p-2})-k_1-\cdots-k_{p-1}} Z_{p-1} = D^{r(n_0+\cdots+n_{p-1})-k_1-\cdots-k_{p-1}}(uy_{rn_p})$$
$$= \sum_{k_p \geqslant 0} \binom{r(n_0 + n_1 + \cdots + n_{p-1}) - k_1 - \cdots - k_{p-1}}{k_p} u_{k_p} y_{r(n_0+\cdots+n_p)-k_1-\cdots-k_p}$$

where the sum runs through $0 \leqslant k_p \leqslant r(n_0 + n_1 + \cdots + n_{p-1}) - k_1 - \cdots - k_{p-1}$. Here $n_0 + n_1 + \cdots + n_p = n - p$ and we introduce $k_0 := r(n - p) - k_1 - \cdots - k_p$. Hence

$$L^n y = y_{rn} + \sum_{p=1}^{n} \sum_{\substack{n_0, n_1, \ldots, n_p \geqslant 0 \\ n_0 + n_1, + \cdots + n_p = n-p}} \sum_{k_1, k_2, \ldots, k_p \geqslant 0} \binom{rn_0}{k_1} \binom{r(n_0+n_1) - k_1}{k_2}$$
$$\times \binom{r(n_0+n_1+n_2) - k_1 - k_2}{k_3} \cdots \binom{r(n_0 + \cdots + n_p) - k_1 - \cdots - k_{p-1}}{k_p}$$
$$\times u_{k_1} u_{k_2} \ldots u_{k_p} y_{k_0}. \qquad (12)$$

Let us introduce the new summation indices

$$m_0 = n_0, m_1 = n_0 + n_1, m_2 = n_0 + n_1 + n_2, \ldots, m_p = n_0 + n_1 + \cdots + n_p.$$

They are then subject to

$$0 \leqslant m_0 \leqslant m_1 \leqslant m_2 \leqslant \cdots \leqslant m_p = n - p$$

and only to these conditions. Finally, the summation over $m_0, m_1, \ldots, m_p$ and the summation over $k_1, k_2, \ldots, k_p$ can be interchanged and we arrive at

$$L^n y = y_{rn} + \sum_{p=1}^{n} \sum_{k_0, k_1, \ldots, k_p \geqslant 0} C_{k_1 k_2 \ldots k_p} u_{k_1} u_{k_2} \ldots u_{k_p} y_{k_0} \qquad (13)$$

where the coefficients $C_{k_1 k_2 \ldots k_p}$ have the accounced form (11). □

Let us present some of the terms in theorem 1 more explicitly. There holds

$$L^n y = y_{nr} + \sum_{k=0}^{(n-1)r} C_k u_k y_{(n-1)r-k} + \sum_{k_1, k_2 = 0}^{(n-2)r} C_{k_1 k_2} u_{k_1} u_{k_2} y_{(n-2)r - k_1 - k_2} + \cdots + u^n y$$

where the intermediate dots indicate the terms of degree greater than 2 and less than n in u and where

$$C_k = \sum_{m=0}^{n-1} \binom{rm}{k} \equiv \binom{0}{k} + \binom{r}{k} + \cdots + \binom{(n-1)r}{k}$$
$$C_{k_1 k_2} = \sum_{m_1=0}^{n-2} \sum_{m_2=0}^{m_1} \binom{rm_2}{k_1} \binom{rm_1 - k_1}{k_2}.$$

Note that for $p = n$ we have $k_0 = k_1 = \cdots = k_n = 0$ and the single coefficient $C_{00\ldots0} = 1$. The terms of the inner sum of (10) are in one-to-one correspondence with number-theoretical partitions. More precisely, in the general non-commutative case the terms correspond to vectors $(k_0, k_1, \ldots k_p)$ of integers $k_0, k_1, \ldots, k_p$ such that

$$k_0, k_1, \ldots, k_p \geqslant 0 \qquad \text{and} \qquad k_0 + k_1 + \cdots + k_p = r(n-p).$$

Proposition 3. The inner sum over $k_0, k_1, \ldots, k_p$ in (10) has exactly $\binom{r(n-p)+p}{p}$ terms.

Proof. The number of ordered partitions of an integer m into s non-negative integers equals, according to textbooks of combinatories, $\binom{m+s-1}{s-1}$ [4, 5]. Here we insert $m = r(n-p)$, $s = p + 1$. □

In the special commutative case some terms can be added together to form a new term. The new terms correspond to unordered partition of $r(n-p)$ into $p + 1$ non-negative integers. Unfortunately, there is no practical closed formula for the number of unordered partitions. Clearly, the total number of terms on the right-hand side of equation (10) equals

$$\sum_{p=0}^{n} \binom{r(n-p)+p}{p}.$$

Let us specialize our result to $r = 2$, that means present an explicit formula for the powers L^n of the one-dimensional Schrödinger operator $L = D^2 + u$. We find

$$L^n = (D^2 + u)^n = D^{2n} + \sum_{p=1}^{n} \sum_{k_0,k_1,\ldots,k_p} C_{k_1 k_2 \ldots k_p} u_{k_1} u_{k_2} \ldots u_{k_p} D^{k_0} \tag{14}$$

where

$$C_{k_1 k_2 \ldots k_p} = \sum_{m_0,m_1,\ldots,m_p} \binom{2m_0}{k_1} \binom{2m_1 - k_1}{k_2} \binom{2m_2 - k_1 - k_2}{k_3} \cdots \binom{2m_p - k_1 - \cdots - k_{p-1}}{k_p}.$$

The non-negative summation indices are subject to $k_0 + k_1 + \cdots + k_p = 2(n - p)$ and $m_0 \leqslant m_1 \leqslant \cdots \leqslant m_p = n - p$.

3. An alternative formula for L^n

Schimming [6, 7] derived an explicit formula for the right-hand sides of the KdV hierarchy (5) or (6). He used a special symbolism in order to present a non-recursive expression for the differential polynomials $G_n[u]$ ($n = 1, 2, \ldots$). This method also works in our case. Namely, let us abbreviate, as before, the kth derivative of $u = u(x)$ by $u_k = u^{(k)}$ ($k = 0, 1, 2, \ldots$) and let us supplement this by the formal definitions

$$u_{-1} = 0, \ldots, u_{1-r} = 0 \qquad u_{-r} = 1.$$

Furthermore, we set

$$c(k, l) := \binom{k}{l} + \delta_l^{k+r}$$

where the second term is a Kronecker symbol.

Theorem 2. There holds

$$L^n y = \sum_{k_1,k_2,\ldots,k_n} \left(\prod_{i=1}^{n} c(k_{i-1}, k_i) u_{k_{i-1}-k_i} \right) y_{k_n} \tag{15}$$

where the sum runs through the integers $k_1, k_2, \ldots, k_n$ such that

$$0 \leqslant k_i \leqslant k_{i-1} + r \qquad \text{for } i = 1, 2, \ldots, n, \ k_0 = 0.$$

Proof. Let us set here $y_{(n)} := L^n y$. Then

$$y_{n+1} = L y_{(n)} = y_{(n)}^{(r)} + u y_{(n)}$$

$$y_{(n+1)}^{(k)} = y_{(n)}^{(k+r)} + \sum_{l=0}^{k} \binom{k}{l} u_{k-l} y_{(n)}^{(l)} = \sum_{l=0}^{k+r} c(k, l) u_{k-l} y_{(n)}^{(l)}.$$

Thus, we have some linear recursion equation for the double sequence $(y_{(n)}^{(k)})$ $(n, k = 0, 1, 2, \ldots)$. This recursion can be solved by means of mathematical induction. The first two steps read

$$\begin{aligned} y_{(n)}^{(k)} &= \sum_{k_1=0}^{k+r} c(k, k_1) u_{k-k_1} y_{(n-1)}^{(k_1)} \\ &= \sum_{k_1=0}^{k+r} \sum_{k_2=0}^{k_1+r} c(k, k_1) c(k_1, k_2) u_{k-k_1} u_{k_1-k_2} y_{(n-2)}^{(k_2)} \end{aligned}$$

and the procedure stops after n steps:

$$y_{(n)}^{(k_0)} = \sum_{k_1=0}^{k_0+r} \ldots \sum_{k_n=0}^{k_{n-1}+r} c(k_0, k_1) \ldots c(k_{n-1}, k_n) u_{k_0-k_1} \ldots u_{k_n-k_{n-1}} y_{(0)}^{(k_n)}.$$

The result follows by some rearrangement and by setting $k_0 = 0$, $y_{(0)}^{(k_n)} = y_{k_n}$, $y_{(n)}^{(0)} = L^n y$. Formula (14) is not as simple as it appears to be, because each coefficient $c(k_{i-1}, k_i)$ involves two or three cases:

$$c(k, l) = \binom{k}{l} \text{ for } l \leqslant k \qquad c(k, l) = 1 \text{ for } l = k + r \qquad c(k, l) = 0 \text{ otherwise.}$$

We can rewrite (14) in the form

$$L^n y = \sum_{k=0}^{rn} P_{(n)}^k [u] y_k$$

where

$$P_{(n)}^k [u] = \sum_{k_1, k_2, \ldots, k_{n-1}} \left(\prod_{i=1}^{n} c(k_{i-1}, k_1) u_{k_{i-1}-k_i} \right) \qquad k_0 = 0, \; k_n = k.$$

□

The following might be seen as a curiosity; we mention it for the sake of completeness. Define an infinite matrix $U = (u_{kl})$ for $k, l = 0, 1, 2, \ldots$ by

$$u_{kl} = \binom{k}{l} u_{k-1} + \delta_l^{k+r}.$$

Denote the elements of $U^n = UU \ldots U$ (n times) by $u_{kl}^{(n)}$. Then

$$P_{(n)}^k [u] = u_{0k}^{(n)}.$$

4. Discussion

Torriani [3] discussed the one-dimensional Schrödinger operator $L = D^2 + u$ in the scalar case. He presented recursion formulae with respect to n for the coefficients $c_{k_1, k_2 \ldots k_p}^{(n)}$ in

$$L^n y = \sum_{p=0}^{n} \sum_{k_1, k_2, \ldots k_p} c_{k_1, k_2, \ldots, k_p}^{(n)} u_{k_1} \ldots u_{k_2} \ldots u_{k_p} y_{k_0}.$$

We are not able to generalize Torriani's recursion to the general case $L = D^r + u$; we think that our explicit formulae are more useful. Schimming and Rida presented [8] an explicit formula for the powers L^n of a first-order operator $L = D + u$ in terms of the Bell polynomials $B_n (n = 1, 2, \ldots)$. Namely, we showed

$$L^n = \sum_{k=0}^{n} \binom{n}{k} B_k(u, u', \ldots, u^{(n-1)}) D^{n-k}$$

where $B_0 \equiv 1$, $B_1 \equiv y_1$, and

$$B_n(y_1, y_2, \ldots, y_n) = \sum_{d=1}^{n} \sum_{n_2,\ldots,n_d=1}^{n} \binom{n-1}{n_2} \binom{n_2-1}{n_3} \cdots \binom{n_{d-1}-1}{n_d} y_{n_d} y_{n_{d-1}-n_d} \cdots y_{n_2-n_3} y_{n-n_2}$$

for $n \geqslant 2$, cf [8] for details.

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References

[1] Lax P 1968 Integrals of nonlinear equations of evolution and solitary waves *Commun. Pure Appl. Math.* **21** 467–90

[2] Gelfand I M and Dickii L A 1976 Fractional powers of operators and Hamiltonian systems *Funct. Anal. Appl.* **10** 259–73

[3] Torriani H H 1987 Integral powers of the Sturm–Liouville operator and the theory of partitions *Adv. Appl. Math.* **8** 345–53

[4] Riordan J 1958 *Combinatorial Indentities* (New York: Wiley)

[5] Riordan J 1978 *An Introduction to Combinatorial Analysis* (Princeton, NJ: Princeton University Press)

[6] Schimming R 1988 An explicit expression for the Kortweg–de Vries hierarchy *Z. f. Anal. Anw.* **7** 203–14

[7] Schimming R 1995 An explicit expression for the Korteweg–de Vries hierarchy *Acta Applicandae Mathematicase* **39** 489–505

[8] Schimming R and Rida S Z 1996 Non-commutative Bell polynomials *Int. J. Algebra Comput.* **6** 635–44